

# UNIMODAL SEQUENCES SHOW LAMBERT W IS BERNSTEIN

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ABSTRACT. We consider a sequence of polynomials appearing in expressions for the derivatives of the Lambert  $W$  function. The coefficients of each polynomial are shown to form a positive sequence that is log-concave and unimodal. This property implies that the positive real branch of the Lambert  $W$  function is a Bernstein function.

## 1. INTRODUCTION

The Lambert  $W$  function was defined and studied in [5]. It is a multivalued function having branches  $W_k(z)$ , each of which obeys  $W_k \exp(W_k) = z$ . The principal branch  $W_0$  maps the set of positive reals to itself, and is the only branch considered here. Therefore we omit the subscript 0 for brevity. The  $n$ th derivative of  $W$  is given implicitly by

$$(1.1) \quad \frac{d^n W(x)}{dx^n} = \frac{\exp(-nW(x))p_n(W(x))}{(1+W(x))^{2n-1}} \quad \text{for } n \geq 1 ,$$

where the polynomials  $p_n(w)$  satisfy  $p_1(w) = 1$ , and the recurrence relation

$$(1.2) \quad p_{n+1}(w) = -(nw + 3n - 1)p_n(w) + (1+w)p'_n(w) \quad \text{for } n \geq 1 .$$

In [6], the first 5 polynomials were printed explicitly:

$$\begin{aligned} p_1(w) &= 1 , \quad p_2(w) = -2 - w , \quad p_3(w) = 9 + 8w + 2w^2 , \\ p_4(w) &= -64 - 79w - 36w^2 - 6w^3 , \\ p_5(w) &= 625 + 974w + 622w^2 + 192w^3 + 24w^4 . \end{aligned}$$

These initial cases suggest the conjecture that each polynomial  $(-1)^{n-1}p_n(w)$  has all positive coefficients, and if this is true, then  $dW(x)/dx$  is a completely monotonic function [11]. We here prove the conjecture and prove in addition that the coefficients are unimodal and log-concave.

## 2. FORMULAE FOR THE COEFFICIENTS

In view of the conjecture, we write

$$(2.1) \quad p_n(w) = (-1)^{n-1} \sum_{k=0}^{n-1} \beta_{n,k} w^k .$$

We now give several theorems regarding the coefficients.

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**Theorem 2.1.** *The coefficients  $\beta_{n,k}$  defined in (2.1) obey the recurrence relations*

$$(2.2) \quad \beta_{n,0} = n^{n-1}, \quad \beta_{n,1} = 3n^n - (n+1)^n - n^{n-1},$$

$$(2.3) \quad \beta_{n,n-1} = (n-1)!, \quad \beta_{n,n-2} = (2n-2)(n-1)!,$$

$$(2.4) \quad \beta_{n+1,k} = (3n-k-1)\beta_{n,k} + n\beta_{n,k-1} - (k+1)\beta_{n,k+1}, \quad 2 \leq k \leq n-3.$$

*Proof.* By substituting (2.1) into (1.2) and equating coefficients.  $\square$

**Theorem 2.2.** *An explicit expression for the coefficients  $\beta_{n,k}$  is*

$$(2.5) \quad \beta_{n,k} = \sum_{m=0}^k \frac{1}{m!} \binom{2n-1}{k-m} \sum_{q=0}^m \binom{m}{q} (-1)^q (q+n)^{m+n-1}.$$

*Proof.* We rewrite (1.1) in the form

$$p_n(W(x)) = (1+W(x))^{2n-1} e^{nW(x)} \frac{d^n W(x)}{dx^n}.$$

From the Taylor series of  $W(x)$  around  $x=0$ , given in [5], we obtain

$$\frac{d^n W(x)}{dx^n} = \sum_{m=n}^{\infty} \frac{(-m)^{m-1}}{(m-n)!} x^{m-n}.$$

Substituting this into the expression of  $p_n$ , using  $x = We^W$  and changing the index of summation, we obtain the equation

$$(2.6) \quad p_n(w) = (1+w)^{2n-1} \sum_{s=0}^{\infty} (-1)^{n+s-1} (n+s)^{n+s-1} \frac{w^s}{s!} e^{(n+s)w}.$$

We expand the right side around  $w=0$  and equate coefficients of  $w$ .  $\square$

**Remark 2.3.** The polynomials  $p_n(w)$  can be expressed in terms of the diagonal Poisson transform  $\mathbf{D}_n[f_s; z]$  defined in [10], namely, by (2.6)

$$(2.7) \quad p_n(w) = (-1)^{n-1} (1+w)^{2(n-1)} \mathbf{D}_n[(n+s)^{n-1}; -w].$$

**Theorem 2.4.** *The coefficients can equivalently be expressed either in terms of shifted  $r$ -Stirling numbers of the second kind  $\left\{ \begin{smallmatrix} n+r \\ m+r \end{smallmatrix} \right\}_r$  defined in [3],*

$$(2.8) \quad \beta_{n,k} = \sum_{m=0}^k (-1)^m \binom{2n-1}{k-m} \left\{ \begin{smallmatrix} 2n-1+m \\ n+m \end{smallmatrix} \right\}_n,$$

*or in terms of Bernoulli polynomials of higher order  $B_n^{(z)}(\lambda)$  defined in [9],*

$$(2.9) \quad \beta_{n,k} = \sum_{m=0}^k (-1)^m \binom{2n-1}{k-m} \binom{m+n-1}{n-1} B_{n-1}^{(-m)}(n),$$

*or in terms of the forward difference operator  $\Delta$  [7, p. 188],*

$$\beta_{n,k} = \sum_{m=0}^k \binom{2n-1}{k-m} \frac{(-1)^m}{m!} \Delta^m n^{m+n-1}.$$

*Proof.* We convert (2.5) using identities found in [3] and [8] respectively.

$$\left\{ \begin{matrix} n+r \\ m+r \end{matrix} \right\}_r = \frac{1}{m!} \sum_{q=0}^m (-1)^{m-q} \binom{m}{q} (q+r)^n$$

and

$$B_n^{(-m)}(r) = \frac{n!}{(m+n)!} \sum_{q=0}^m (-1)^{m-q} \binom{m}{q} (q+r)^{m+n}.$$

□

### 3. PROPERTIES OF THE COEFFICIENTS

We now give theorems regarding the properties of the  $\beta_{n,k}$ . We recall the following definitions [12]. A sequence  $c_0, c_1 \dots c_n$  of real numbers is said to be *unimodal* if for some  $0 \leq j \leq n$  we have  $c_0 \leq c_1 \leq \dots \leq c_j \geq c_{j+1} \geq \dots \geq c_n$ , and it is said to be *logarithmically concave* (or log-concave for short) if  $c_{k-1}c_{k+1} \leq c_k^2$  for all  $1 \leq k \leq n-1$ . We prove that for each fixed  $n$ , the  $\beta_{n,k}$  are unimodal and log-concave with respect to  $k$ . Since a log-concave sequence of positive terms is unimodal [15], it is convenient to start with the log-concavity property.

**Theorem 3.1.** *For fixed  $n \geq 3$  the sequence  $\{k! \beta_{n,k}\}_{k=0}^{n-1}$  is log-concave.*

*Proof.* Using (2.5) we can write

$$k! \beta_{n,k} = (2n-1)! \sum_{m=0}^k \binom{k}{m} x_m y_{k-m},$$

where

$$(3.1) \quad x_m = \sum_{j=0}^m \binom{m}{j} a_j, \quad a_j = (-1)^j (n+j)^{m+n-1},$$

and  $y_m = 1/(2n-1-m)!$ . Since the binomial convolution preserves the log-concavity property [13, 14], it is sufficient to show that the sequences  $\{x_m\}$  and  $\{y_m\}$  are log-concave. We have

$$\begin{aligned} a_{j-1}a_{j+1} &= (-1)^{j-1}(n+j-1)^{m+n-1}(-1)^{j+1}(n+j+1)^{m+n-1} \\ &= (-1)^{2j}((n+j)^2-1)^{m+n-1} < (-1)^{2j}(n+j)^{2(m+n-1)} = a_j^2. \end{aligned}$$

Thus the sequence  $\{a_j\}$  is log-concave and so is  $\{x_m\}$  due to (3.1) and the aforementioned property of the binomial convolution. The sequence  $\{y_m\}$  is log-concave because

$$\begin{aligned} y_{m-1}y_{m+1} &= \frac{1}{(2n-1-m+1)!} \frac{1}{(2n-1-m-1)!} \\ &= \frac{2n-1-m}{2n-1-m+1} \frac{1}{(2n-1-m)!} \frac{1}{(2n-1-m)!} < y_m^2. \end{aligned}$$

□

Now we prove that the coefficients  $\beta_{n,k}$  are positive. The following two lemmas are useful.

**Lemma 3.2.** *If a positive sequence  $\{k!c_k\}_{k \geq 0}$  is log-concave, then*

- (i)  $\{(k+1)c_{k+1}/c_k\}$  is non-increasing;

- (ii)  $\{c_k\}$  is log-concave;
- (iii) the terms  $c_k$  satisfy

$$(3.2) \quad c_k c_m \geq \binom{k+m}{k} c_0 c_{k+m} \quad (0 \leq m \leq k+1) .$$

*Proof.* The statements (i) and (ii) are obvious. To prove (iii) we apply a method used in [1]. Specifically, by (i) we have for  $0 \leq p \leq k$

$$\frac{c_{p+1}}{c_p} \geq \frac{k+p+1}{p+1} \frac{c_{k+p+1}}{c_k} .$$

Apply the last inequality for  $p = 0, 1, 2, \dots, m$  with  $m \leq k+1$ , and form the products of all left-hand and right-hand sides. As a result, after the cancellation we obtain

$$\frac{c_m}{c_0} \geq \frac{k+1}{1} \frac{k+2}{2} \dots \frac{k+m}{m} \frac{c_{k+m}}{c_k} ,$$

which is equivalent to (3.2).  $\square$

**Lemma 3.3.** *If the coefficients  $\beta_{n,k}$  are positive, then for fixed  $n \geq 3$  they satisfy*

$$(3.3) \quad \frac{(k+1)\beta_{n,k+1}}{\beta_{n,k}} < n-1 .$$

*Proof.* By Theorem 3.1 and under the assumption of lemma, for fixed  $n \geq 3$  the sequence  $\{k!\beta_{n,k}\}_{k=0}^{n-1}$  meet the conditions of Lemma 3.2. Applying the inequality (3.2) with  $m = 1$  to this sequence gives  $(k+1)\beta_{n,k+1}/\beta_{n,k} \leq \beta_{n,1}/\beta_{n,0}$ . Then the lemma follows as due to (2.2)

$$\frac{\beta_{n,1}}{\beta_{n,0}} = \frac{3n^n - (n+1)^n - n^{n-1}}{n^{n-1}} = 3n - n \left(1 + \frac{1}{n}\right)^n - 1 < 3n - 2n - 1 = n - 1 .$$

$\square$

**Theorem 3.4.** *The coefficients  $\beta_{n,k}$  are positive.*

*Proof.* We prove the statement by induction on  $n$ . It is true for  $n \leq 5$  (see §1). Assume that for some fixed  $n$  all the members of the sequence  $\{\beta_{n,k}\}_{k=0}^{n-1}$  are positive. Since  $\beta_{n+1,0} = (n+1)^n > 0$  and  $\beta_{n+1,n} = n! > 0$  by (2.2) and (2.3), we only need to consider  $k = 1, 2, \dots, n-1$ .

Substituting inequalities  $\beta_{n,k+1} < (n-1)\beta_{n,k}/(k+1)$  and  $\beta_{n,k-1} > k\beta_{n,k}/(n-1)$ , which follow from (3.3), in the recurrence (2.4) immediately gives the result

$$\beta_{n+1,k} > (3n-k-1)\beta_{n,k} + n \frac{k}{n-1} \beta_{n,k} - (k+1) \frac{n-1}{k+1} \beta_{n,k} = \left(2n + \frac{k}{n-1}\right) \beta_{n,k} > 0 .$$

Thus the proof by induction is complete.  $\square$

**Corollary 3.5.** *The sequence  $\{\beta_{n,k}\}_{k=0}^{n-1}$  is unimodal for  $n \geq 3$ .*

*Proof.* By Theorem 3.4 the sequence  $\{\beta_{n,k}\}_{k=0}^{n-1}$  is positive, therefore by Theorem 3.1 and Lemma 3.2(ii) it is log-concave and, hence, unimodal.  $\square$

## 4. RELATION TO CARLITZ'S NUMBERS

There is a relation between the coefficients  $\beta_{n,k}$  and numbers  $B(\kappa, j, \lambda)$  introduced by Carlitz in [4]. Comparing the formulae (2.8) and (5.1) with the corresponding [4, eq.(6.3)] and [4, eq.(2.9)], taking into account that he uses the notation  $R(n, m, r) = \left\{ \begin{smallmatrix} n+r \\ m+r \end{smallmatrix} \right\}_r$ , we find

$$(4.1) \quad \beta_{n,k} = (-1)^k B(n-1, n-1-k, n) .$$

It follows that for  $n \geq 3$ , the sequence  $\{B(n-1, k, n)\}_{k=0}^{n-1}$  is log-concave together with  $\{\beta_{n,k}\}_{k=0}^{n-1}$ .

Using the property [4, eq.(2.7)] that  $\sum_{j=0}^{\kappa} B(\kappa, j, \lambda) = (2\kappa-1)!!$ , we can compute  $p_n(w)$  at the singular point where  $W = -1$  (cf. (1.1)). Thus, substituting  $w = -1$  in (2.1) gives  $p_n(-1) = (-1)^{n-1}(2n-3)!!$ . Thus  $w = -1$  is not a zero of  $p_n(w)$ .

We also note that the numbers  $B(\kappa, j, \lambda)$  are polynomials of  $\lambda$  and satisfy a three-term recurrence [4, eq. (2.4)]

$$(4.2) \quad B(\kappa, j, \lambda) = (\kappa + j - \lambda)B(\kappa-1, j, \lambda) + (\kappa - j + \lambda)B(\kappa-1, j-1, \lambda)$$

with  $B(\kappa, 0, \lambda) = (1-\lambda)^{\bar{\kappa}}$ ,  $B(0, j, \lambda) = \delta_{j,0}$ . This gives one more way to compute the coefficients  $\beta_{n,k}$ , specifically, for given  $n$  and  $k$  we find a polynomial  $B(n-1, n-1-k, \lambda)$  using (4.2) and then set  $\lambda = n$  to use (4.1).

## 5. CONCLUDING REMARKS

It has been established that the coefficients of the polynomials  $(-1)^{n-1}p_n(w)$  are positive, unimodal and log-concave. These properties imply an important property of  $W$ . In particular, it follows from formula (1.1) and Theorem 3.4 that  $(-1)^{n-1}(dW/dx)^{(n-1)} > 0$  for  $n \geq 1$ . Since  $W(x)$  is positive for all positive  $x$  [5], this means that the derivative  $W'$  is completely monotonic and  $W$  itself is a Bernstein function [2].

Some additional identities can be obtained from the results above. For example, computing  $\beta_{n,n-1}$  by (2.8) and comparing with (2.3) gives

$$\sum_{m=0}^{n-1} (-1)^m \binom{2n-1}{n-m-1} \left\{ \begin{smallmatrix} 2n-1+m \\ n+m \end{smallmatrix} \right\}_n = (n-1)! .$$

A relation between  $\left\{ \begin{smallmatrix} 2n-1+m \\ n+m \end{smallmatrix} \right\}_n$  and  $B_{n-1}^{(-m)}(n)$  can be obtained from (2.8) and (2.9), but this is a special case of [4, eq. (7.5)]. It is finally interesting to note that (2.8) and (2.9) can be inverted. Indeed, in these formulae for fixed  $n$ , the sequence  $(-1)^k \beta_{n,k}$  is a convolution of two sequences and the corresponding relation between their generating functions is  $G(w) = (1-w)^{2n-1}F(w)$ . Since  $F(w) = G(w)(1-w)^{-(2n-1)} = G(w) \sum_{k \geq 0} \binom{2n-2+k}{2n-2} w^k$ , the inverse of, for example, (2.8) is

$$(5.1) \quad \left\{ \begin{smallmatrix} 2n-1+m \\ n+m \end{smallmatrix} \right\}_n = \sum_{k=0}^{n-1} (-1)^k \beta_{n,k} \binom{2n-2+m-k}{2n-2} .$$

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